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q-analogues of Verma representations of quantum algebras for q, a root of unity: the case of $U_q(sl(3))$*

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Abstract. Verma representation theory for classical Lie algebra is extended to study the representation of quantum universal enveloping algebra (quantum algebra) for the non-generic case that q is a root of unity. On certain subspaces and quotient spaces of the Verma space, finite- and infinite-dimensional irreducible or indecomposable representations of $sl_q(3) = U_q(sl(3))$ are obtained in explicit matrix forms.

1. Introduction

At present, the quantum algebra $U_q(L)$ (a q-analogue of universal enveloping algebra of a classical Lie algebra L) is an important topic in mathematical physics [1-3]. This is because of its crucial role in nonlinear integrable systems of physics through the Yang-Baxter equation (YBE) [4, 5]. The representation theory of quantum algebra is progressing rapidly from different directions [6-12]. Although the q-deformed Boson realization [10-12] is a useful method to construct explicit matrices of representations for quantum algebras, it is only powerful enough for symmetric representations of quantum algebra $(A_l)_q = U_q(sl(l+1))$ and $(C_l)q = U_q(sp(2l))$. In order to obtain the representations with another symmetry, we have studied the regular representation with $(A_l)_q$ as an example [13], which is closely related to Vera's theory [14] for Lie algebra.

In this paper we will generally consider an extension of Verma's theory for the quantum algebra with $sl_q(3)$ as an illustration. Since the discussion of the generic case that q is not a root of unity is only a q-deformation of the Lie algebra case (for the study of Lie algebra A_2, see [15] and [16]), we will mainly pay attention to the non-generic case that q is a root of unity, i.e. $q^p = 1$ ($p = 3, 4, 5 \ldots$).

We first describe the main results and some technical details in this paper as follows. In section 2, through a quite lengthy calculation and by induction, we write down some q-deformed commutation relations (q-relations) among the bases of $sl_q(3)$. These bases are chosen by taking the q-analogue of the Poincaré-Birkhoff-Witt (PBW) theorem [8, 18] for the quantum algebra into account. In section 3, we explicitly construct an

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infinite-dimensional representation (quantum Verma representation) on the so-called quantum Verma module with a lower weight by making use of the obtained q-relations. In order to get the finite-dimensional representations, which are necessary for constructing solutions of the YBE, two distinct classes of invariant subspaces are indentified by some extreme vectors on the quantum Verma space in section 4. As the induced transformations of the quantum Verma representation on the corresponding quotient spaces, the finite-dimensional representations are constructed explicitly in both the generic case in section 5 and the non-generic case in section 6. In the former case, the obtained finite representations are either irreducible or completely reducible. In the latter case, the extreme vectors are defined by the non-generic condition $q^p=1$ $(p=3,5,\ldots)$ and our construction leads to the finite-dimensional indecomposable (reducible, but not completely reducible) representations.

Finally, it is pointed out that these new representations can be used to construct non-generic R-matrices [17] for YBE through the universal R-matrix of $\mathrm{sl}_q(3)$ [18].

2. The q-deformed commutation relations and bases for $\mathrm{sl}_q(3)$

The quantum algebra $\mathrm{sl}_q(3)$ is an associative over the complex number field $\mathbb{C}$ and has generators $E_i\equiv E_i^+$, $F_i\equiv E_i^-$ and $H_i (i=1,2)$ that satisfy the basic q-deformed commutation relations

$$\begin{aligned} &[H_1, E_1^{\pm}]=\pm 2E_1^{\pm} && [H_1, E_2^{\pm}]=\mp E_2^{\pm} \\ &[H_2, E_1^{\pm}]=\mp E_1^{\pm} && [H_2, E_2^{\pm}]=\pm 2E_2^{\pm} \\ &[E_i, F_j]=\delta_{ij}[H_i] && [H_i, H_j]=0 \qquad i,j=1,2 \end{aligned} \tag{2.1}$$

and the Serre relations

$$E_i^{\pm 2}E_{i\pm 1}^{\pm}-(q+q^{-1})E_i^{\pm}E_{i\pm 1}^{\pm}E_i^{\pm}+E_{i\pm 1}^{\pm}E_i^{\pm}=0 \tag{2.2}$$

where $[f]=(q^f-q^{-f})/(q-q^{-1})$ is defined for any operator and number f and $q\in\mathbb{C}$.

When $q\to 1$, (2.1) just are the commutation relations satisfied by the Chevalley basis of classical Lie algebra $A_2=\mathrm{su}(3)$, and $\{E_i, F_i\}$ corresponds to the simple roots $\alpha_1=e_1-e_2$ and $\alpha_2=e_2-e_3$, when $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$. So we need to find the third pair $\{E_3, F_3\}$ corresponding to the third positive root $\alpha_3=\alpha_1+\alpha_2=e_1-e_3$ for the construction of the basis of $\mathrm{sl}_q(3)$. According to Rosso [8] and Burroughs [18],

$$E_3=E_1E_2-qE_2E_1 \qquad F_3=F_1F_2-qF_2F_1. \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} &E_1E_3=q^{-1}E_3E_1 && E_2E_3=qE_3E_2 \\ &F_1E_3=E_3F_1+E_2K_1 && F_2E_3=E_3F_2-qE_1K_2^{-1} \\ &K_1E_3=qE_3K_1 && K_2^{-1}E_3=q^{-1}E_3K_2^{-1} \end{aligned} \tag{2.4}$$

where $K_1=q^{H_1}$ and $K_2=q^{H_2}$. By induction we prove

$$\begin{aligned} &E_2E_1^m=q^{-m}E_1^mE_2-q^{-1}[m]E_1^{m-1}E_3 \\ &E_3E_1^n=q^nE_1^nE_3 \qquad E_3E_2^n=q^{-n}E_2^nE_3 \\ &F_1E_3^n=E_3^nF_1+[n]E_2E_3^{n-1}K_1 \qquad F_2E_3^n=E_3^nF_2-q[n]E_1E_3^{n-1}K_2^{-1} \\ &F_iE_i^n=E_i^nF_i-(q-q^{-1})^{-1}E_i^{n-1}(q^{n-1}K_i-q^{1-n}K_i^{-1}) \qquad i=1,2. \end{aligned} \tag{2.5}$$

Using (2.1), (2.4) and (2.5), we can easily arrange the basis for $\mathrm{sl}_q(3)$ as

$$E_1^{m_1}E_2^{m_2}E_3^{m_3}F_1^{n_1}F_2^{n_2}F_3^{n_3}H_1^{s_1}H_2^{s_2} \tag{2.6}$$

(where m_i, n_i and $s_j \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, $i = 1, 2, 3$; $j = 1, 2$) because we may commute any of the generators E_i, F_i and H_j. This is just a special case of the q-analogue of PBW theorem for $\mathrm{sl}_q(l+1)$ proved by Rosso [8], which is a generalization of the PBW theorem for Lie algebra.

3. Quantum Verma module for $\mathrm{sl}_q(3)$

Because an associative algebra itself is a linear space, the left transformation L: $\mathrm{sl}_q(3) \to \mathrm{End}(\mathrm{sl}_q(3))$ defined by

$$L(g)X = g \cdot X \qquad \forall g, X \in \mathrm{sl}_q(3)$$

determines an infinite-dimensional representation of $\mathrm{sl}_q(3)$, which is called the left regular representation. As $q \to 1$, it becomes the master representation of su(3) [15], which is a subrepresentation obtained by constraining the regular representation of su(3)-universal enveloping algebra in its subalgebra su(3).

Let $\mathscr{H}$ be the Cartan subalgebra of $\mathrm{sl}_q(3)$, which is generated by H_1 and H_2. If λ $\lambda \in \mathscr{H}^*$, i.e. λ is a linear function on $\mathscr{H}$, then $\{F_i,\ H_j - \lambda(H_j)\mathbb{1} | i = 1, 2, 3;\ j = 1, 2,\ \mathbb{1} = X(0, \dots, 0)\}$ generate a left ideal $I(\lambda) = \mathrm{sl}_q(3)\ (\Sigma_{i=1}^3 F_i + \Sigma_{j=1}^2 (H_j - \lambda(H_j)))$, which is a left invariant subspace of $\mathrm{sl}_q(3)$. The corresponding quotient space

$$\begin{aligned} V(\lambda) &\equiv V(\lambda_1, \lambda_2) \\ &= \mathrm{sl}_q(3)/I(\lambda)(\lambda_i \equiv \lambda(H_i)) \colon \{f_\lambda(m, n, k) = E_1^m E_2^n E_3^k \bmod I(\lambda),\ m, n, k \in \mathbb{Z}^+\} \end{aligned}$$

with the action of $\mathrm{sl}_q(3)$ induced by L is called the quantum Verma module (a q-analogue of the Verma module for classical Lie algebra). When $q \to 1$, it becomes the usual Verma module, an indecomposable standard cyclic module with the lowest weight λ: (λ_1, λ_2) [19]. Here, $|\lambda\rangle = f_\lambda(0, 0, 0)$ is such an extreme vector that

$$H_i|\lambda\rangle = \lambda_i|\lambda\rangle \qquad F_i|\lambda\rangle = 0 \qquad i = 1, 2.$$

Using (2.5) we explicitly write the representation $\rho^{[\lambda]}$ of $\mathrm{sl}_q(3)$ on the quantum Verma space $V(\lambda)$ as follows:

$$\begin{aligned} &H_1 f_\lambda(m, n, k) = (2m - n + k + \lambda_1) f_\lambda(m, n, k) \\ &H_2 f_\lambda(m, n, k) = (2n - m + k + \lambda_2) f_\lambda(m, n, k) \\ &E_1 f_\lambda(m, n, k) = f_\lambda(m+1, n, k) \\ &F_1 f_\lambda(m, n, k) = q^{\lambda_1}[k] f_\lambda(m, n+1, k-1) - [m][m - 1 - n + k + \lambda_1] f_\lambda(m-1, n, k) \\ &E_2 f_\lambda(m, n, k) = q^{-m} f_\lambda(m, n+1, k) - q^{-n-1}[m] f_\lambda(m-1, n, k+1) \\ &F_2 f_\lambda(m, n, k) = q^k [n][1 - \lambda_2 - n] f_\lambda(m, n-1, k) - q^{1-n-\lambda_2}[k] f_\lambda(m+1, n, k-1). \end{aligned} \tag{3.1}$$

According to Rosso [8], some theorems about representations of classical Lie algebras can be directly generalized to the quantum algebra in the generaric case. So, when λ is a dominant integral function, i.e. $\lambda(H_i) = \lambda_i \in \mathbb{Z}^- = \{-1, -2, \dots\}$, the Verma representation $\rho^{[\lambda]}$ may induce some finite-dimensional representations on certain quotient spaces. These representations, which are irreducible for the generic case, are no longer irreducible for the non-generic cases. This is because $[\alpha p]=0$ when $q^p=1$ and $\alpha\in\mathbb{Z}^+$ and some new extreme vectors result from $[\alpha p]=0$. We need to point out that what we study here is the quantum Verma module with the lowest weight and the discussion about the quantum Verma module with the highest weight is just parallel to the former discussion. In particular because of the symmetry of weights under the Weyl group, the finite-dimensional representations resulting from the quantum Verma representation with highest weight are equivalent to those resulting from the quantum Verma representation with lowest weight when q is not a root of unity. For the above reasons we only need to study the case with lowest weight.

4. Two classes of invariant subspaces

In this section we will determine two classes of $\rho^{[\lambda]}$-invariant subspaces, on which $\rho^{[\lambda]}$ subduces new representations.

4.1. *The first class of invariant subspaces*

Invariant subspaces of the first class,

$$S_v=\mathrm{sl}_q(3)\cdot v\colon \{E_1^m E_2^n E_3^k v | m, n, k\in\mathbb{Z}^+\}$$

are standard cyclic modules [19] defined by such extreme weight vectors v that

$$\rho^{[\lambda]}(F_i)v=0 \qquad H_i v=M(H_i)v\equiv M_i v \qquad i=1,2 \tag{4.1}$$

where $M(\in\mathscr{H}^*)$ is a weight function so that $M_1-\lambda_1\geqslant 0$ or $M_2-\lambda_2\geqslant 0$ for $M_1=\lambda_1$. The lowest weights of these standard cyclic modules are M: (M_1, M_2).

The weight space $V[M_1, M_2]\equiv V_{\alpha\beta}(\alpha,\beta\in\mathbb{Z})$ with weight (M_1, M_2) can be labelled by two indices α and β:

$$\alpha=\tfrac{1}{3}(2M_1+M_2-2\lambda_1-\lambda_2) \qquad \beta=\tfrac{1}{3}(2M_2+M_1-2\lambda_2-\lambda_1).$$

It is easy to see that $V_{\alpha\beta}=V[M_1, M_2]$ is spanned by the weight vectors

$$\{f_\lambda(\alpha-k, \beta-k, k) | k=0,1,2,\ldots,\min(\alpha,\beta)\}$$

where $\min(\alpha,\beta)=\alpha$, if $\alpha\leqslant\beta$, and $\min(\alpha,\beta)=\beta$, if $\alpha>\beta$.

Let

$$v=\sum_{k=0}^{\min(\alpha,\beta)} C_k f(\alpha-k,\beta-k,k)\in V_{\alpha\beta} \tag{4.2}$$

be an extreme vector satisfying (4.1). Then the equations $\rho^{[\lambda]}(F_1)v=0$ and $\rho^{[\lambda]}(F_2)v=0$ respectively give

$$\begin{aligned} &q^{\lambda_1}[k+1]C_{k+1}-[\alpha-k][\alpha-\beta+k+\lambda_1-1]C_k=0\\ &q^{2-\beta+k-\lambda_2}[k+1]C_{k+1}-q^k[\beta-k][1-\beta+k-\lambda_2]C_k=0.\end{aligned} \tag{4.3}$$

Because the recurrence relations (4.3) determine some extremal vector v, they must be identical and thereby α and β must be chosen carefully. Then,

$$\frac{q^{2-\beta+k-\lambda}[k+1]}{q^{\lambda_1}[k+1]}=\frac{[\beta-k][1-\beta+k-\lambda_2]q^k}{[\alpha-k][\alpha-\beta+k+\lambda_1-1]}$$

i.e.

$$1+q^{-2(1-\lambda_2)}(1-q^{2(\beta-k)})=q^{2(\alpha-\beta)}+q^{2(1-\lambda_1)}(q^{-2\alpha}-q^{-2k}). \tag{4.4}$$

For three well-defined solutions of (4.4)

$$\begin{array}{lll} \text{(I)} & k=\alpha=0 & \beta=1-\lambda_2 \\ \text{(II)} & k=\beta=0 & \alpha=1-\lambda_1 \\ \text{(III)} & \alpha=\beta=2-\lambda_1-\lambda_2 & \end{array} \tag{4.5}$$

we get three extreme vectors

$$\begin{aligned} &v_1=f_\lambda(0,1-\lambda_2,0) \qquad v_2=f_\lambda(1-\lambda_1,0,0) \\ &v_3=\sum_{k=3-\lambda_1}^{-2-\lambda_1-\lambda_2}\left(\prod_{i=3-\lambda_1}^{k}([i]^{-1}q^{-\lambda_1}[3-\lambda_1-\lambda_2-i][i+\lambda_1-2])\right) \\ &\qquad \times f_\lambda(2-\lambda_1-\lambda_2-k,2-\lambda_1-\lambda_2-k,k) \end{aligned} \tag{4.6}$$

where $C_0\in\mathbb{C}$, v_1, v_2 and v_3 respectively possess weights

$$M(1)=(\lambda_1+\lambda_2-1,2-\lambda_2) \qquad M(2)=(2-\lambda_1,\lambda_1+\lambda_2-1)$$

$$M(3)=(2-\lambda_2,2-\lambda_1).$$

The corresponding subspaces $S(v_1)$, $S(v_2)$ and $S(v_3)$ are denoted by S_1, S_2 and S_3 respectively. It needs to be pointed out that S_3 is ill-defined by (4.6) for the non-generic case because $[\alpha p]=0$ $(\alpha\in\mathbb{Z}^+)$.

4.2. *The second class of invariant subspaces*

For the non-generic case that q is a root of unity, it follows from $[\alpha p]=0$ $(\alpha\in\mathbb{Z})$ that

$$\begin{aligned} &F_1f_\lambda(\alpha p,n,\alpha_3p)=0 \\ &F_2f_\lambda(m,\alpha_2p,\alpha_3p)=0 \qquad \text{for } \alpha_1,\alpha_2,\alpha_3\in\mathbb{Z}^+ \end{aligned} \tag{4.7}$$

that is to say, $f_\lambda(\alpha_1p,\alpha_2p,\alpha_3p)$ are extreme vectors that satisfy

$$\begin{aligned} &F_if_\lambda(\alpha_1p,\alpha_2p,\alpha_3p)=0 \qquad i=1,2 \\ &H_1f_\lambda(\alpha_1p,\alpha_2p,\alpha_3p)=[(2\alpha_1-\alpha_2+\alpha_3)p+\lambda_1]f_\lambda(\alpha_1p,\alpha_2p,\alpha_3p) \\ &H_2f_\lambda(\alpha_1p,\alpha_2p,\alpha_3p)=[(2\alpha_2-\alpha_1+\alpha_3)+\lambda_2]f_\lambda(\alpha_1p,\alpha_2p,\alpha_3p). \end{aligned} \tag{4.8}$$

For given α_1, α_2 and α_3 the extreme vector f_λ $(\alpha_1p,\alpha_2p,\alpha_3p)$ defines an invariant subspace $S[\boldsymbol{\alpha}]=S(\alpha_1,\alpha_2,\alpha_3)$: $\{f_\lambda(m,n,k)|m\geqslant\alpha,p,\ n\geqslant\alpha_2p,\ k\geqslant\alpha_3p\}$.

In this non-generic case, though the invariant subspaces S_1, S_2 and S_3 are still invariant, they are no longer irreducible for some situations. For example, if $S[\boldsymbol{\alpha}]\subset S_i$, then $S[\boldsymbol{\alpha}]$ is an invariant subspace of S_i. We will discuss the latter in detail.

5. Representations on quotient spaces for the generic case

From Verma theory we know that the quotient module of a maximal proper submodule is finite-dimensional and irreducible so long as the highest (or lowest) weight is the dominant integral weight. This conclusion can be generalized to quantum algebra, but we must distinguish two cases, generic and non-generic. For the generic case, the conclusion is the same; for the non-generic case, the dominant integral weight results in a finite-dimensional representation, but it is not irreducible for some situations. In this section, we only discuss the generic case. This is the basis for the discussion of non-generic case in the next section.

In contrast to the results or classical Lie algebra for λ_1 and $\lambda_2 \in \mathbb{Z}^-$, the subspace $S_{12} = S_1 + S_2$ is a unique maximal proper submodule generated by $E_2^{1-\lambda_2}$ and $E_1^{1-\lambda_1}$ and thus the quotient space $\Omega(\lambda) = V(\lambda)/S_{12}$ is a finite-dimensional $\rho^{[\lambda]}$-invariant subspace. On this space, $\rho^{[\lambda]}$ induces a finite-dimensional representation, which is irreducible for the generic case.

Now we consider the basis vectors for $\Omega(\lambda_1, \lambda_2) \equiv \Omega(\lambda)$. Define

$$\bar{f}_\lambda(m, n, k) = f_\lambda(m, n, k) \bmod (S_1 + S_2)$$

then $\bar{f}_\lambda(1-\lambda_1, 0, 0) = \bar{f}_\lambda(0, 1-\lambda_2, 0) = 0$. Because

$$E_2^n E_1^m = \sum_{s=0}^{n} C_s(m, n) E_3^s E_1^{m-s} E_2^{n-s} \tag{5.1}$$

there are some constraints among the vectors $\bar{f}_\lambda(m, n, k)$,

$$\bar{f}_\lambda(1-\lambda_1, n, k) = -q^{mn} \sum_{s=1}^{n} C_s(1-\lambda_1, n) q^{s(m-n)} f_\lambda(1-\lambda_1 - s, n-s, s+k) \tag{5.2}$$

where $C_s(m, n)$ is given by the following recurrence relations:

$$\begin{aligned} &C_0(m, n) = q^{-mn} \\ &C_{n+1}(m, n+1) = q^{2(n-m-1)}[m-n] C_n(m, n) \\ &C_s(m, n+1) = q^{2s-m}[C_s(m, n) - q^{-2}[m+1-s] C_{s-1}(m, n)]. \end{aligned} \tag{5.3}$$

Just as for the case of Lie algebra A_2[15], the above constraints result in complicated expressions for representations on $\Omega(\lambda)$. Therefore, we only study the representations induced by $\rho^{[\lambda]}$ in an explicit form for some special case.

It is observed from (3.1) that the subspace $J(\lambda)$: $\{f_\lambda(m, 1-\lambda_2+n, k) | m, n, k \in \mathbb{Z}^+\}$ is an invariant subspace. This is because the action of $\rho^{[\lambda]}$ does not change a vector $f_\lambda(m, n, k)(n \geq 1-\lambda_2)$ into $f_\lambda(m, n', k)(k' < 1-\lambda_2)$. On the quotient space $Q(\lambda) = V(\lambda)/J(\lambda)$: $\{F_\lambda(m, n, k) = f_\lambda(m, n, k) \bmod J(\lambda),\ m, \in \mathbb{Z}^+,\ n = 0, 1, \ldots, -\lambda_2\}$, the representations $\rho^{[\lambda]}$ induces a representation

$$\begin{aligned} &H_1 F_\lambda(m, n, k) = (2m - n + k + \lambda_1) F_\lambda(m, n, k) \\ &H_2 F_\lambda(m, n, k) = (2n - m + k + \lambda_2) F_\lambda(m, n, k) \\ &E_1 F_\lambda(m, n, k) = F_\lambda(m+1, n, k) \\ &F_1 F_\lambda(m, n, k) = \theta(-\lambda_2 - n - 1) q^{\lambda_1}[k] F_\lambda(m, n+1, k-1) - [m] \\ &\qquad \times [m-1-n+k+\lambda_1] F_\lambda(m-1, n, k) \\ &E_2 F_\lambda(m, n, k) = \theta(-\lambda_2 - 1 - n) q^{-m} F_\lambda(m, n+1, k) - q^{-n-1}[m] F_\lambda(m-1, n, k+1) \\ &F_2 F_\lambda(m, n, k) = q^k [n][1-\lambda_2 - n] F_\lambda(m, n-1, k) - q^{1-\lambda_2-n}[k] F_\lambda(m+1, n, k-1) \end{aligned} \tag{5.4}$$

where

$$\theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0. \end{cases}$$

For the special case of $\lambda_2=0$, the representation (5.4) is rewritten as follows:

$$\begin{aligned}
&H_1F_\lambda(m,k)=(2m+k+\lambda_1)F_\lambda(m,k)\\
&H_2F_\lambda(m,k)=(k-m+\lambda_2)F_\lambda(m,k)\\
&E_1F_\lambda(m,k)=F_\lambda(m+1,k)\\
&F_1F_\lambda(m,k)=-[m][m-1+\lambda_1+k]F_\lambda[m-1,k)\\
&E_2F_\lambda(m,k)=-q^{-1}[m]F_\lambda(m-1,k+1)\\
&F_2F_\lambda(m,k)=-q[k]F_\lambda(m+1,k-1)
\end{aligned}\tag{5.5}$$

where $F_\lambda(m,k)=F_\lambda(m,0,k)$.

Define the subspace $W(l)$ ($l\in\mathbb{Z}^+$): $\{F_\lambda(m,k)|m+k=1\}$. Because (5.5) result in

$$H_1W(l), H_2W(l), E_2W(l), F_2W(l)\subset W(l)$$

$$E_1W(l)\subset W(l+1)\qquad E_1W(1-\lambda_1)=\{0\}$$

the subspace

$$S(\lambda_1)=\sum_{l=1-\lambda_1}^{\infty}W(l)$$

is invariant and its quotient space $\pi(\lambda_1)=Q(\lambda_1,\lambda_2=0)/S(\lambda_1)$: $\{F(m,k|\lambda_1)=F_\lambda(m,k)$ mod $S(\lambda_1)|0\leqslant m+k\leqslant-\lambda_1\}$ is finite-dimensional. On $\pi(\lambda_1)$ (5.5) induces a finite-dimensional representation

$$\begin{aligned}
&H_1F(m,k|\lambda_1)=(2m+k+\lambda_1)F(m,k|\lambda_1)\\
&H_2F(m,k|\lambda_1)=(k-m)F(m,k|\lambda_1)\\
&E_1F(m,k|\lambda_1)=\theta(-1-\lambda_1-m-k)F(m+1,k|\lambda_1)\\
&F_1F(m,k|\lambda_1)=-[m][m+1+\lambda_1+k]F(m-1,k|\lambda_1)\\
&E_2F(m,k|\lambda_1)=-q^{-1}[m]F(m-1,k+1|\lambda_1)\\
&F_2F(m,k|\lambda_1)=-q[k]F(m+1,k-1|\lambda_1)
\end{aligned}\tag{5.6}$$

with the dimension

$$\dim\pi(\lambda_1)=\tfrac{1}{2}(1-\lambda_1)(2-\lambda_1).\tag{5.7}$$

6. Representations for the non-generic case

6.1. The representations induced by $\rho^{[\lambda]}$

We notice that $S[\boldsymbol{\alpha}]$ is an invariant subspace. On its quotient space $\Omega[\alpha_1,\alpha_2,\alpha_3]=V(\lambda)/S[\boldsymbol{\alpha}]$: $\{\nu(m,n,k)=f_\lambda(m,n,k)$ mod $S[\boldsymbol{\alpha}]|0\leqslant m\leqslant\alpha_1p-1,\ 0\leqslant n\leqslant\alpha_2p-1,\ 0\leqslant k\leqslant\alpha_3p-1\}$. $\rho^{[\lambda]}$ induces a finite-dimensional representation

$$\begin{aligned}
&H_1\nu(m,n,k)=(2m-n+k+\lambda_1)\nu(m,n,k)\\
&H_2\nu(m,n,k)=(2n-m+k+\lambda_2)\nu(m,n,k)\\
&E_1\nu(m,n,k)=\theta(\alpha_1p-2-m)\nu(m+1,n,k)\\
&F_1\nu(m,n,k)=\theta(\alpha_2p-2-n)q^\lambda[k]\nu(m,n+1,k-1)-[m]\\
&\qquad\times[m-1-n+k+\lambda_1]\nu(m-1,n,k)
\end{aligned}\tag{6.1}$$

$$E_2\nu(m,n,k)=\theta(\alpha_2p-2-n)q^{-m}\nu(m,n+1,k)$$
$$\qquad -q^{-n-1}[m]\theta(\alpha_3p-1-k)\nu(m-1,n,k+1)$$
$$F_2\nu(m,n,k)=q^k[n][1-\lambda_2-n]\nu(m,n-1,k)-q^{1-n-\lambda_2}\theta(\alpha_1p-2-m)\nu(m+1\ n,k-1)$$

with the dimension

$$\dim\Omega[\alpha_1,\alpha_2,\alpha_3]=\alpha_1\alpha_2\alpha_3p.$$

Now, we prove that the representation (6.1) is indecomposable (reducible, but not completely reducible), if there is an $\alpha_i\geqslant 2$. In fact, when $i=1$, $\Omega[\alpha_1\alpha_2\alpha_3]$ has an invariant subspace $W_1:\{\nu(m,n,k)|(\alpha_1-1)p\leqslant m\leqslant\alpha_1p-1,\ 0\leqslant n\leqslant\alpha_2p-1,\ 0\leqslant k\leqslant\alpha_3p-1\}$ with dimension $d=(p-1)\alpha_2\alpha_3p^2$. If there exists an invariant subspace $\bar{W}_1$ complementary to W, i.e. $\bar{W}_1\oplus W_1=\Omega[\alpha_1\alpha_2\alpha_3]$, then there is a vector

$$\nu=\sum_{m=k}^{\alpha_1p-1}C_m\nu(m,n,k)$$

with $C_k\neq 0$ and $k<(\alpha_1-1)p$. Acting on ν by E_1, we have

$$E_1^{(\alpha_1-1)p-k}\nu=\sum_{m=k}C_m\nu(m+(\alpha_1-1)p-k,n,k)$$
$$=\sum_{m=0}C_{m+k}\nu(m'+(\alpha_1-1)p,n,k)(\neq 0)\in W.$$

Due to invariance of $\bar{W}_1$, $E_1^{(\alpha_1-1)p-k}\nu\in\bar{W}_1$, that is to say, $\bar{W}_1\cap W_1\neq\{0\}$. Then a contradiction appears and so an invariant complementary space for W_1 does not exist.

6.2. *The non-generic structure of the representation (5.6)*

In order to analyse the reductions of representation (5.6) when q is a root of unity, the representation (5.6) is illustrated in figure 1. Each lattice (m,k) in $\triangle$OAB denotes a weight vector $F(m,k|\lambda_1)$ and the arrows in the figure represent the actions of E_i and F_i $(i=1,2)$.

On the character lines l_1: $m=\beta_1p$ and l_2: $k=\beta_2p$ ($\beta_1,\beta_2\in\mathbb{Z}^+$ and $\leqslant-\lambda_1/p$), the lattices (β_1p,k) and (m,β_2p) correspond to extreme vectors $F(\beta_1p,k|\lambda_1)$ and $F(m,\beta_2p|\lambda_2)$ respectively. In fact, it follows from (5.6) that

$$F_1F(\beta_1p,k|\lambda_1)=0\qquad F_2F(m,\beta_2p|\lambda_1)=0\qquad E_2F(\beta_1p,k|\lambda_2)=0.$$

The extreme vectors $F(\beta_1p,k|\lambda_1)$ and $F(m,\beta_2p|\lambda_2)$ define the invariant subspaces $U(\beta_1)$: $\{F(m,n|\lambda_1)\in\pi(\lambda_1)|m\geqslant\beta_1p\}$ and $M(\beta_2)$: $\{F(m,n|\lambda_1)\in\pi(\lambda_1)|n\geqslant\beta_2p\}$,

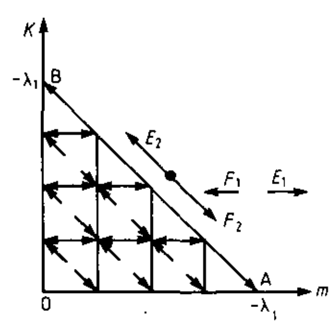

Figure 1.

respectively. $U(\beta_1)$ corresponds to $\triangle$ACD and $M(\beta_2)$ corresponds to $\triangle$BEF, as shown in figure 2.

If $U(\beta_1)\cap M(\beta_2)\neq\{0\}$, i.e. $(\beta_1+\beta_2)p\leqslant-\lambda_1$, then $U(\beta_1)\cap M(\beta_2)$ is a smaller invariant subspace and representation (5.6) induces a representation with lower dimension (see figure 3).

Now we see an example with $-\lambda_1=-3p$ and $p=3$. On the ten-dimensional invariant subspace $U(1)\cap M(2)$:

$$\{F(3,3|9),\ F(4,3|9),\ F(5,3|9),\ F(6,3|9),\ F(3,4|9),\ F(4,4|9),\ F(5,4|9),$$
$$F(3,5|9),\ F(4,5|9),\ F(3,6|9)\}$$

the representation (5.6 induces new representations

$$H_1=18e_{11}+20e_{22}+22e_{33}+24e_{44}+19e_{55}+21e_{66}+23e_{77}+20e_{88}+22e_{99}+21e_{10\,10}$$

$$H_2=3(e_{11}+e_{22}+e_{33}+e_{44})+4(e_{55}+e_{66}+e_{77})+5(e_{88}+e_{99})+6e_{10\,10}$$

$$E_1=e_{21}+e_{32}+e_{43}+e_{65}+e_{76}+e_{98}$$

$$F_1=-[2]e_{23}-e_{56}-[2]^2e_{67}-[2]e_{89}$$

$$E_2=-q^{-1}(e_{52}+[2]e_{63}+e_{86}+[2]e_{97}+e_{10\,9})$$

$$F_2=-q(e_{25}+e_{36}+e_{74}+[2]e_{68}+[2]e_{79}).$$

where e_{ij} is a 10×10 matrix unit so that $(e_{ij})_{kl}=\delta_{ik}\delta_{jl}$.

Finally, we point out that the representation (5.6) is indecomposable and its reduction is completely classified in this section, but for the other representations in section 5, the reductions are very complicated and not discussed in this paper. In fact,

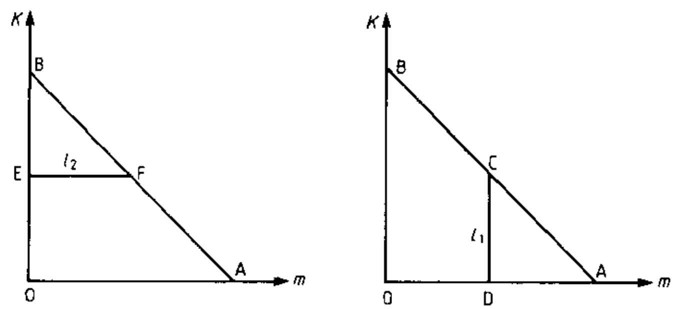

Figure 2.

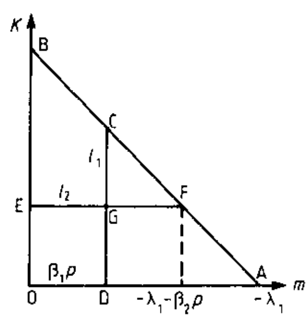

Figure 3.

for the non-generic case without introducing the Lusztig operators [7], further work is needed for the complete classification of reductions for any representation of any quantum algebra.

Note added in proof. After this paper was submitted, we received some preprints, 'representations of quantum group at root of unity' Kac and de Concini, and RIMS (703, 709) by Jimbo *et al*, in which general representation theory in the generic case is fully built from a purely mathematical point of view. However, our main results (for indecomposable representations) are not covered by these works. In particular, we give some explicit constructions that are useful for concrete problems in physics.

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